

Charged BTZ black holes in the context of massive gravity's rainbow

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Regarding the significant interests in thermodynamics of black objects, we examine charged BTZ black holes. We consider massive gravity context with an energy dependent spacetime to enrich the results. In addition, we consider all the constants as energy dependant ones. We investigate thermodynamic properties of the solutions by calculating the heat capacity and free energy. We also analyze thermal stability and study the possibility of Hawking-Page phase transition. At last, we study geometrical thermodynamics of these black holes.

I. INTRODUCTION

Einstein's general relativity (GR) is a successful relativistic theory of gravity at low energies or large distances where the graviton is massless. The basic question is whether there exists a consistent massive extension of such GR as the study of a consistent extension of GR by a mass term is supported by several both theoretical and observational considerations [1]. For instance, the graviton mass improves the cosmological constant problem [1]. Also, the observational consideration suggests that 95% of the universe are dark energy and dark matter [2] which is based on assumption that GR is equally valid at all length scales. So, the modifications of GR by a mass term could possibly change this scenario over large distances. In other words, regarding massive gravitons may be solved some of fundamental problems in gravitation and cosmology. Massive gravity can explain the current acceleration of the universe without considering a cosmological constant [3, 4]. In addition, massive gravitons are candidates for gravitational waves and dark matter [5], and also they product the gravitational waves during the inflation [6]. Furthermore, one can obtain a maximum mass of neutron stars more than $3M_{sun}$ [7] in the context of massive gravity and it can modified the thermodynamical quantities of black holes as well [8–10].

Fierz and Pauli were the first to study the theory describing a free massive graviton [11, 12]. Later, it has been found out that this theory of massive gravity suffers the Boulware-Deser (BD) ghost instability at the non-linear level [13, 14]. Recently, a significant progress has been made towards constructing massive gravity theories without such instability [15]. Furthermore, the nonlinear massive modifications to GR is also studied by many people in various perspectives. Particularly, Refs. [1, 15, 16] study a class of nonlinear massive gravity theories in which the ghost field is absent [17, 18]. The simplest way to construct a massive gravity is to simply add a mass term to the Einstein-Hilbert action, giving the graviton a mass m in such a way that GR is recovered when $m \rightarrow 0$. Since a mass term breaks the diffeomorphism invariance of the theories, hence, the energy momentum is no longer conserved in this class of massive gravity.

On the other hand, the usual energy-momentum relations or dispersion relations in special relativity may be modified with corrections in the order of Planck length by modifying the Lorentz-Poincaré symmetry. This deformed formalism of special relativity is known as “Doubly Special Relativity” [19–22]. Later, Magueijo and Smolin [23] proposed an idea about generalization of this idea in curved spacetime. This formalism is known commonly as gravity's rainbow. The idea of gravity's rainbow formalism is that the free falling observers who make measurements with energy E will observe the same laws of physics as in modified special relativity. In fact, the gravity's rainbow produces a correction to the spacetime metric which becomes significant as soon as the particle's energy/momentum approaches the Planck energy. In this formalism, the connection and curvature depend on energy in such a way that the usual Einstein's equations is replaced by a one parameter family of equations. In this context, the Gauss-Bonnet theory and dilaton gravity are generalized to an energy-dependent Gauss-Bonnet and dilatonic theories of gravity which analyze the black hole solutions in these energy-dependent theory [24, 25]. Recently, the critical behavior of uncharged and charged solutions of black holes in Gauss-Bonnet gravity's rainbow is analyzed and it has found that the generalization to a charged case puts an energy dependent restriction to different parameters [26]. Two classes of $F(R)$ gravity's rainbow

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solutions are also investigated [27]. In first case, the energy dependent $F(R)$ gravity without energy momentum tensor is studied and, secondly, $F(R)$ gravity's rainbow in the presence of conformally invariant Maxwell source is analyzed. In the gravity's rainbow context, the Starobinsky model of inflation is investigated where gravity rainbow functions are written in the power-law form of the Hubble parameter [28]. In this context, the spectral index of curvature perturbation, the tensor-to-scalar ratio and consistency of this models with Planck 2015 data are also discussed. Moreover, Galileon gravity's rainbow by considering Vaidya spacetime has been studied in [29]. Also, the Hawking, fiducial and free-fall temperatures of the black hole in gravity's rainbow have been investigated in Ref. [30]. The absence of black holes at LHC [31], remnants of black objects [32], nonsingular universes in Einstein and Gauss-Bonnet gravities [33, 34] have been analyzed in the gravity's rainbow background. From the astrophysical perspective, it was shown that the existence of energy dependent spacetime can modify the hydrostatic equilibrium equation of stars [35].

To explore the foundations of classical and quantum gravity, GR in three spacetime dimensions has become a very popular model [36]. One of the drawbacks of this GR model in three dimensions was that there were no Newtonian limit [37] and no propagating degrees of freedom. In 1992, Bañados, Teitelboim and Zanelli (BTZ) came with surprising result that three dimensional gravity with a negative cosmological constant has a black hole solution [38]. Contrary to the Schwarzschild and Kerr black holes, BTZ black hole is asymptotically anti-de Sitter (adS) rather than asymptotically flat, and has no curvature singularity at the origin. In fact, BTZ black holes provide a good understanding of certain central issues like black hole thermodynamics [39–41], quantum gravity, string and gauge theory and more importantly the AdS/CFT conjecture [42, 43]. Furthermore, BTZ solutions perform a central role to improve our perception of gravitational interaction in low dimensional spacetime [44]. The charged BTZ black hole is the analogous solution of adS-Maxwell gravity in three dimensions [39, 45, 46]. Recently, thermodynamics and phase structure of the charged black hole solutions in both the grand canonical and canonical ensembles are studied [47–49].

The paper at hand, regards three dimensional charge black holes with three generalizations; energy dependant constants, gravity's rainbow and massive gravity. Recently, three dimensional charged black holes in the presence of the massive gravity have been investigated [49]. Here, we apply the generalization to gravity's rainbow to understand how this generalization would modify previous results. In fact, we would like to see how the energy dependant spacetime would affects thermodynamical structure of the massive charged BTZ black holes. Such generalization is necessary from different aspects; First of all, black holes and their physics are governed by high energy physics. This indicates that it is necessary to include the upper limit of Plank energy on energies that particles can acquire. This is the prescription of the gravity's rainbow. On the other hand, it is stated that quantum corrections of quantum gravity could be observed as energy dependency of the spacetime [50, 51]. In other words, one could include the quantum corrections in form of the energy dependency of the spacetime which leads to gravity's rainbow. These provide us a motivation to consider gravity's rainbow alongside of massive gravity.

The consideration of the energy dependency of the constants is rooted in studies that are conducted in the context of the renormalization group flow [52]. These studies are emphasized on the energy dependency of the constants on the scale of theory probed. Through several studies, the flow of the cosmological [53] and Newton [54] constants were examined. Since the scale measurement of the theory under consideration depends on the energy that probe can acquire, therefore, it is logical to consider all the constant as energy dependent ones. Such consideration has been taken into account in the context of the Gauss-Bonnet gravity and it was shown that it enriches both geometrical and thermodynamical aspects of the black holes [25, 26]. Here too, we employ such consideration to take all the constants energy dependent. Such an idea provides different perspectives for the observers who are at different distance from black holes under consideration. In addition, it would have specific contributions to other studies that could be conducted in the context of these types of the black holes (we refer the reader to Refs. [55, 56] for some examples). Our other motivation for considering such set up for black holes is to provide maximum number of the generalizations. These specific generalizations are effective in specific regions of energy which provide a better picture regarding the physics of the black holes.

Geometrical thermodynamics (GTs) is one of the interesting methods for studying the properties of thermodynamical systems. In this method the Riemannian geometry is used to construct phase space. The Ricci scalar of this phase space is employed to extract some information regarding thermodynamical behavior of the system. In other words, GTs is a bridge between geometry and thermodynamics. This informational Ricci scalar are obtained through its divergencies. These divergencies determine three important points;

- I) Bound points which separate solutions with positive temperature (physical systems) from those with negative temperature (non-physical systems).
- II) Phase transition points which represent discontinuities in thermodynamical quantities such as heat capacity.
- III) The sign of the Ricci scalar around divergence points determines the nature of interaction on molecular level [57].

Weinhold introduced the first geometrical thermodynamical approach in 1975. Weinhold's approach was based on internal energy as thermodynamical potential [58, 59]. Then, Ruppeiner proposed an alternative approach which has entropy as its thermodynamical potential [60, 61]. Since Weinhold and Ruppeiner's approaches are not Legendre

invariant, Quevedo introduced another approach for GTs [62, 63]. Several investigations regarding thermodynamics of the black holes through these methods were done in refs. [64–71].

On the other hand, it was shown that mentioned methods may confront specific problems in describing thermodynamical properties of the black holes (see refs. [72–75], for more details). In other words, obtained results of these three approaches were not consistent with those extracted from other methods. Therefore, in order to remove the shortcomings of the other methods, Hendi et al proposed a new thermodynamical metric (HPEM) [72]. It is notable that, it was shown that employing this new metric leads into consistent results regarding thermodynamical properties of the black holes. We refer the reader to Ref. [76] for a comparative study regarding these four thermodynamical metrics and Ref. [77] regarding the application of HPEM metric in studying critical behavior of the system.

The outline of the paper is as follows. First, we will introduce the basic field equations and metric, and extract black holes solutions. Next, thermodynamical quantities are calculated and the first law of thermodynamics for black holes is examined. Then thermodynamical properties of the black holes are studies through, mass, temperature, heat capacity and free energy. Next, we will study thermodynamics of these black holes in the context of GTs and show the consistency of its results with divergencies and bound points of heat capacity. The paper is finished with some closing remarks.

II. BLACK HOLE SOLUTIONS

The general formalism of gravity's rainbow could be obtained by using a deformation of the standard energy-momentum relation

$$E^2 f^2(\varepsilon) - p^2 g^2(\varepsilon) = m^2, \quad (1)$$

where the dimensionless energy ratio is $\varepsilon = E/E_P$ in which E and E_P are, respectively, the energy of test particle and the Planck energy. Since the energy of a test particle can not exceed the Plank energy, we should remind $0 < \varepsilon \leq 1$. Here, $f(\varepsilon)$ and $g(\varepsilon)$ are energy functions which are restricted with the following condition in infrared limit

$$\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 1, \quad \lim_{\varepsilon \rightarrow 0} g(\varepsilon) = 1. \quad (2)$$

Regarding the analogy between the energy-momentum four vector (E, \vec{p}) with time-space one (t, \vec{x}) , it is possible to use the energy functions to build an energy dependant spacetime with following recipe

$$\hat{g}(\varepsilon) = \eta^{ab} e_a(\varepsilon) \otimes e_b(\varepsilon), \quad (3)$$

where

$$e_0(\varepsilon) = \frac{1}{f(\varepsilon)} \tilde{e}_0, \quad e_i(\varepsilon) = \frac{1}{g(\varepsilon)} \tilde{e}_i, \quad (4)$$

with \tilde{e}_0 and \tilde{e}_i being the energy independent frame fields (the algorithm of (3) may be originate from the analogy between two invariant relations; the energy-momentum relation and the line element invariant).

Now, we take into account a gravitational structure with a matter field to investigate geometrical properties of the system. Our interest is 3-dimensional Einstein-massive gravity with an abelian $U(1)$ gauge field with the following explicit action

$$\mathcal{I} = -\frac{1}{16\pi G(\varepsilon)} \int d^3x \sqrt{-g} \left[\mathcal{R} - 2\Lambda(\varepsilon) - \mathcal{F} + m(\varepsilon)^2 \sum_i^4 c_i(\varepsilon) \mathcal{U}_i(g, f) \right], \quad (5)$$

where \mathcal{R} and \mathcal{F} are, respectively, the scalar curvature and the Lagrangian of Maxwell electrodynamics, $G(\varepsilon)$ is the energy dependent gravitational constant, $\Lambda(\varepsilon)$ is the energy dependent cosmological constant and f is an energy dependent fixed symmetric tensor. In addition, $\mathcal{F} = F_{\mu\nu} F^{\mu\nu}$ is the Maxwell invariant, in which $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Faraday tensor with A_μ as the gauge potential. Also, $c(\varepsilon)_i$'s are some energy dependent constants and \mathcal{U}_i 's are symmetric polynomials of the eigenvalues of the $d \times d$ matrix $\mathcal{K}_\nu^\mu = \sqrt{g^{\mu\alpha} f_{\alpha\nu}}$, which can be written as follows

$$\begin{aligned} \mathcal{U}_1 &= [\mathcal{K}], & \mathcal{U}_2 &= [\mathcal{K}]^2 - [\mathcal{K}^2], & \mathcal{U}_3 &= [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3], \\ \mathcal{U}_4 &= [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^3][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4]. \end{aligned}$$

Taking the action (5) into account and using the variational principle, we obtain the field equations corresponding to the gravitation and gauge fields as

$$R_{\mu\nu} - \left(\frac{R}{2} - \Lambda(\varepsilon)\right) g_{\mu\nu} + G(\varepsilon) \left(\frac{1}{2} g_{\mu\nu} \mathcal{F} - 2L_{\mathcal{F}} F_{\mu\rho} F_{\nu}^{\rho}\right) + m(\varepsilon)^2 \chi_{\mu\nu} = 0, \quad (6)$$

$$\partial_{\mu} (\sqrt{-g} F^{\mu\nu}) = 0, \quad (7)$$

where the massive term is

$$\begin{aligned} \chi_{\mu\nu} = & -\frac{c_1(\varepsilon)}{2} (\mathcal{U}_1 g_{\mu\nu} - \mathcal{K}_{\mu\nu}) - \frac{c_2(\varepsilon)}{2} (\mathcal{U}_2 g_{\mu\nu} - 2\mathcal{U}_1 \mathcal{K}_{\mu\nu} + 2\mathcal{K}_{\mu\nu}^2) - \frac{c_3(\varepsilon)}{2} (\mathcal{U}_3 g_{\mu\nu} - 3\mathcal{U}_2 \mathcal{K}_{\mu\nu} \\ & + 6\mathcal{U}_1 \mathcal{K}_{\mu\nu}^2 - 6\mathcal{K}_{\mu\nu}^3) - \frac{c_4(\varepsilon)}{2} (\mathcal{U}_4 g_{\mu\nu} - 4\mathcal{U}_3 \mathcal{K}_{\mu\nu} + 12\mathcal{U}_2 \mathcal{K}_{\mu\nu}^2 - 24\mathcal{U}_1 \mathcal{K}_{\mu\nu}^3 + 24\mathcal{K}_{\mu\nu}^4). \end{aligned} \quad (8)$$

Since we are interested in the static charged black hole solutions, we consider the metric of 3-dimensional spacetime with the following energy dependent line element

$$ds^2 = -\frac{\psi(r, \varepsilon)}{f(\varepsilon)^2} dt^2 + \frac{1}{g(\varepsilon)^2} \left(\frac{dr^2}{\psi(r, \varepsilon)} + r^2 d\varphi^2 \right), \quad (9)$$

in which $\psi(r, \varepsilon)$ is the metric function of our black holes.

Our main motivation is to obtain massive black holes in the context of gravity's rainbow. This requires specific modifications in the reference metric in form of [48]

$$f_{\mu\nu} = \text{diag} \left(0, 0, \frac{c(\varepsilon)^2}{g(\varepsilon)^2} \right), \quad (10)$$

where $c(\varepsilon)$ is an arbitrary energy dependent positive constant. Using this metric ansatz (10), \mathcal{U}_i 's will be calculated in forms of [47–49]

$$\mathcal{U}_1 = \frac{c(\varepsilon)}{r}, \quad \mathcal{U}_2 = \mathcal{U}_3 = \mathcal{U}_4 = 0. \quad (11)$$

In order to have a radial electric field, we consider the following gauge potential

$$A_{\mu} = h(r, \varepsilon) \delta_{\mu}^t, \quad (12)$$

where by using the metric (9) with the Maxwell field equation (7), one can find the following differential equation

$$h'(r, \varepsilon) + r h''(r, \varepsilon) = 0, \quad (13)$$

in which the prime and double prime are representing the first and second derivatives with respect to r , respectively. It is a matter of calculation to solve Eq. (13), yielding

$$h(r, \varepsilon) = q(\varepsilon) \ln \left[\frac{r}{l(\varepsilon)} \right], \quad (14)$$

where $q(\varepsilon)$ is an energy dependent integration constant related to the electric charge and $l(\varepsilon)$ is an arbitrary energy dependent constant with length dimension which is considered for the sake of having dimensionless logarithmic argument. It is worthwhile to mention that the corresponding electromagnetic field tensor is $F_{tr} = \frac{q(\varepsilon)}{r}$, which is independent of $l(\varepsilon)$.

In order to obtain metric function, $\psi(r, \varepsilon)$, we use Eq. (9) with Eq. (6), and obtain the following differential equations

$$r g(\varepsilon)^2 \psi'(r, \varepsilon) + 2r^2 \Lambda(\varepsilon) + 2f(\varepsilon)^2 q(\varepsilon)^2 - m(\varepsilon)^2 c(\varepsilon) c_1(\varepsilon) r = 0, \quad (15)$$

$$\frac{r^2}{2} g(\varepsilon)^2 \psi''(r, \varepsilon) + \Lambda(\varepsilon) r^2 - g(\varepsilon)^2 q(\varepsilon)^2 = 0, \quad (16)$$

which correspond to tt (or rr) and $\varphi\varphi$ components of Eq. (6), respectively. It is straightforward to show that metric function is obtained as

$$\psi(r, \varepsilon) = -\frac{\Lambda(\varepsilon) r^2}{g(\varepsilon)^2} - m_0(\varepsilon) - 2G(\varepsilon) f(\varepsilon)^2 q(\varepsilon)^2 \ln \left[\frac{r}{l(\varepsilon)} \right] + \frac{m(\varepsilon)^2 c(\varepsilon) c_1(\varepsilon) r}{g(\varepsilon)^2}, \quad (17)$$

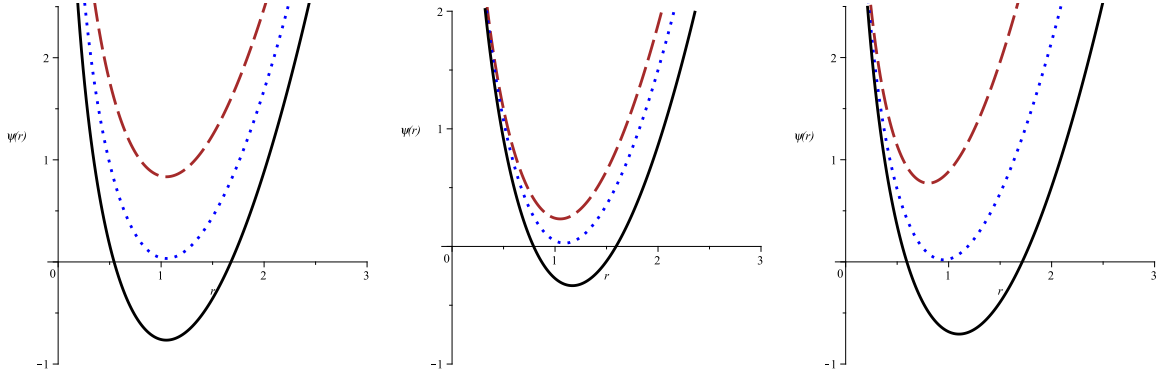


FIG. 1: $\psi(r, \varepsilon)$ versus r for $\Lambda(\varepsilon) = -1$, $G(\varepsilon) = l(\varepsilon) = 1$, $q(\varepsilon) = 1.5$, $g(\varepsilon) = 1$, $f(\varepsilon) = 0.8$ and $m(\varepsilon) = 0.8$. Left panel: $c(\varepsilon) = c_1(\varepsilon) = 1$, $m_0(\varepsilon) = 0.8$ (dashed line), $m_0(\varepsilon) = 1.6$ (dotted line) and $m(\varepsilon) = 2.4$ (continuous line). Middle panel: $m_0(\varepsilon) = 1.4$, $c(\varepsilon) = 1$, $c_1(\varepsilon) = 1$ (dashed line), $c_1(\varepsilon) = 0.7$ (dotted line) and $c_1(\varepsilon) = 0.2$ (continuous line). Right panel: $m_0(\varepsilon) = 2.1$, $c_1(\varepsilon) = 1$, $c(\varepsilon) = 3.1$ (dashed line), $c(\varepsilon) = 1.76$ (dotted line) and $c(\varepsilon) = 0.65$ (continuous line).

where $m_0(\varepsilon)$ is an energy dependant integration constant related to the total mass of black holes. It is worthwhile to mention that the resulting metric function (17) satisfies all the components of field equation (6), simultaneously. In the absence of massive parameter (i.e. $m(\varepsilon) = 0$), the metric function Eq. (17) will be reduced to

$$\psi(r, \varepsilon) = -\frac{\Lambda(\varepsilon) r^2}{g(\varepsilon)^2} - m_0(\varepsilon) - 2G(\varepsilon) f(\varepsilon)^2 q(\varepsilon)^2 \ln\left[\frac{r}{l(\varepsilon)}\right]. \quad (18)$$

Our next step is examination of the geometrical structure of solutions. First, we should look for the existence of essential singularity(ies). The Ricci and Kretschmann scalars of the solutions are, respectively,

$$R = 6\Lambda(\varepsilon) + \frac{2G(\varepsilon) f(\varepsilon)^2 q(\varepsilon)^2}{r^2} - \frac{2m(\varepsilon)^2 c(\varepsilon) c_1(\varepsilon)}{r}, \quad (19)$$

$$R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = 12\Lambda(\varepsilon)^2 - \frac{8\Lambda(\varepsilon) m(\varepsilon)^2 c(\varepsilon) c_1(\varepsilon)}{r} + \frac{2 \left[m(\varepsilon)^4 c(\varepsilon)^2 c_1(\varepsilon)^2 + 4G(\varepsilon) g(\varepsilon)^2 f(\varepsilon)^2 \Lambda(\varepsilon) q(\varepsilon)^2 \right]}{r^2} \\ - \frac{8G(\varepsilon) g(\varepsilon)^2 f(\varepsilon)^2 q(\varepsilon)^2 m(\varepsilon)^2 c(\varepsilon) c_1(\varepsilon)}{r^3} + \frac{12G(\varepsilon)^2 g(\varepsilon)^4 f(\varepsilon)^4 q(\varepsilon)^4}{r^4}. \quad (20)$$

These relations confirm that there is an essential curvature singularity at $r = 0$. For the limit of $r \rightarrow \infty$, the Ricci and Kretschmann scalars yield the values $6\Lambda(\varepsilon)$ and $12\Lambda(\varepsilon)^2$, respectively, which show that for $\Lambda(\varepsilon) > 0$ ($\Lambda(\varepsilon) < 0$), the asymptotical behavior of the solution is (a)dS with an energy dependent cosmological constant.

Our final step in this section is investigation of other geometrical properties such as the existence of regular horizon. For this purpose, we have plotted Figs. 1 and 2 to find the real positive roots of metric function. Evidently, depending on the choices of different parameters, it is possible to observe, two horizons, one extreme horizon and without horizon (naked singularity) for these solutions (see Figs. 1 and 2 for more details). This confirms that the singularity can be covered with an event horizon, and therefore, our solutions are basically representing black holes.

A. Thermodynamics

Now, we intend to calculate the conserved and thermodynamic quantities of the solutions and examine the validity of the first law of thermodynamics.

Using the standard definition of the Hawking temperature with its relation to the surface gravity on the outer horizon r_+ , we obtain

$$T = -\frac{\Lambda(\varepsilon) r_+}{2\pi f(\varepsilon) g(\varepsilon)} + \frac{m(\varepsilon)^2 c(\varepsilon) c_1(\varepsilon)}{4\pi f(\varepsilon) g(\varepsilon)} - \frac{f(\varepsilon) g(\varepsilon) G(\varepsilon) q(\varepsilon)^2}{2\pi r_+}. \quad (21)$$

Furthermore, calculating the flux of electric field at infinity and using the Gauss's law, we can compute the electric charge, Q , as

$$Q = \frac{1}{2} f(\varepsilon) G(\varepsilon) q(\varepsilon). \quad (22)$$

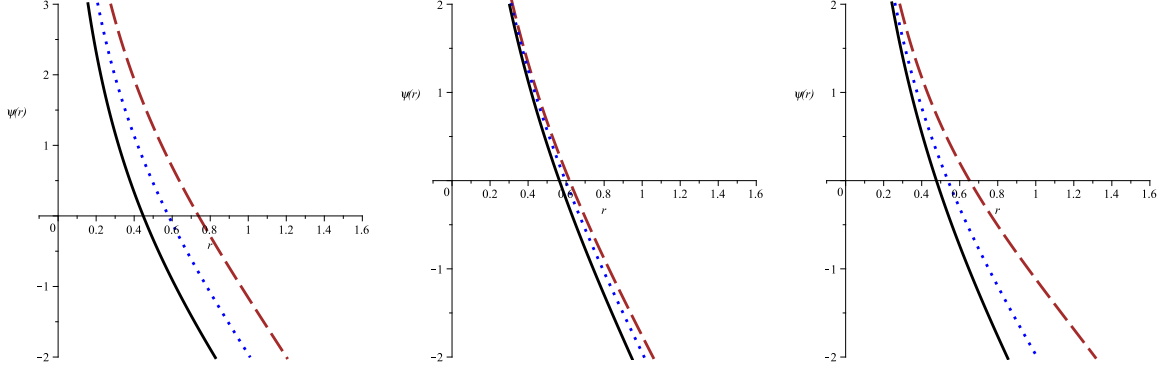


FIG. 2: $\psi(r, \varepsilon)$ versus r for $\Lambda(\varepsilon) = 1$, $G(\varepsilon) = l(\varepsilon) = 1$, $q(\varepsilon) = 1.5$, $g(\varepsilon) = 1$, $f(\varepsilon) = 0.8$ and $m(\varepsilon) = 0.8$.
 Left panel: $c(\varepsilon) = c_1(\varepsilon) = 1$, $m_0(\varepsilon) = 0.8$ (dashed line), $m_0(\varepsilon) = 1.6$ (dotted line) and $m(\varepsilon) = 2.4$ (continuous line).
 Middle panel: $m_0(\varepsilon) = 1.4$, $c(\varepsilon) = 1$, $c_1(\varepsilon) = 1$ (dashed line), $c_1(\varepsilon) = 0.7$ (dotted line) and $c_1(\varepsilon) = 0.2$ (continuous line).
 Right panel: $m_0(\varepsilon) = 2.1$, $c_1(\varepsilon) = 1$, $c(\varepsilon) = 3.1$ (dashed line), $c(\varepsilon) = 1.76$ (dotted line) and $c(\varepsilon) = 0.65$ (continuous line).

Since we are working in Einstein gravity, the entropy of black holes can be obtained by employing the area law. According to this law, we can derive the entropy as a quarter of event horizon area [79–84]

$$S = \frac{\pi}{2g(\varepsilon)} r_+. \quad (23)$$

Also we can obtain the total mass of solutions by using the Hamiltonian approach and/or the counterterm method with the following explicit form

$$M = \frac{m_0(\varepsilon)}{8f(\varepsilon)}, \quad (24)$$

where $m_0(\varepsilon)$ can be computed from the metric function (17) on the horizon ($\psi(r = r_+, \varepsilon) = 0$), and consequently, it may be presented with the following expression

$$m_0(\varepsilon) = -\frac{\Lambda(\varepsilon)r_+^2}{g(\varepsilon)^2} - \frac{2f(\varepsilon)^2q(\varepsilon)^2}{g(\varepsilon)^2} \ln\left(\frac{r_+}{l(\varepsilon)}\right) + \frac{m(\varepsilon)^2c(\varepsilon)c_1(\varepsilon)r_+}{g(\varepsilon)^2}. \quad (25)$$

The electric potential, U , is calculated through the difference of gauge potential between the reference and the horizon, and it is given as

$$U = A_\mu \chi^\mu|_{r \rightarrow reference} - A_\mu \chi^\mu|_{r \rightarrow r_+} = -q(\varepsilon) \ln\left(\frac{r_+}{l(\varepsilon)}\right). \quad (26)$$

Now, we are in a position to check the validity of the first law of thermodynamics. Exploiting thermodynamic quantities such as electric charge (22), entropy (23) and mass (24), with the first law of black hole thermodynamics

$$dM = TdS + UdQ, \quad (27)$$

one can define the intensive parameters conjugate to S and Q . These quantities are the temperature and the electric potential

$$T = \left(\frac{\partial M}{\partial S}\right)_Q \quad \text{and} \quad U = \left(\frac{\partial M}{\partial Q}\right)_S. \quad (28)$$

Using the obtained electric charge (22) and entropy (23) with total mass of the black holes (24), one can find following Smarr-type formula

$$M(S, Q) = -\frac{\Lambda(\varepsilon)S^2}{2f(\varepsilon)\pi^2} + \frac{m(\varepsilon)^2Sc(\varepsilon)c_1(\varepsilon)}{4g(\varepsilon)f(\varepsilon)\pi} - \frac{Q^2}{G(\varepsilon)f(\varepsilon)} \ln\left(2\frac{Sg(\varepsilon)}{\pi l(\varepsilon)}\right). \quad (29)$$

It is a matter of calculation to show that the calculated temperature and electric potential by using Eq. (28) are same as those calculated for the temperature (21) and the electric potential (26). In other word, although massive term and gravity's rainbow modify some of thermodynamic quantities, the first law of thermodynamics is still valid.

Our next thermodynamical quantity of the interest is the heat capacity. This quantity contains information regarding the phase transition points and conditions for thermal stability. The stability of solutions is governed by the sign of heat capacity; its positivity indicates thermal stability while the opposite represents instability. The phase transition points are extracted by finding the divergencies of the heat capacity. In other words, divergencies of the heat capacity may be characterized with a second order phase transitions. In addition, since the roots of heat capacity and temperature are the same (due to form of the heat capacity), the roots of heat capacity are denoted as bound points (separating positive/negative temperature from each other). In particular, we will analyze the heat capacity with fixed charge (in canonical ensemble) and with fixed chemical potential (in grand canonical ensemble).

1. Canonical ensemble

Let us begin this subsection by computing the free energy of the system which provides information regarding the amount of the work that a thermodynamical system can perform. In total, this quantity is given by removing the amount of energy that can not be used to perform work from total internal energy of the system. The unusable energy is a combination of the total entropy and temperature of system. Therefore, for these black holes (in a canonical ensemble with a fixed charge Q), we have the following Helmholtz free energy

$$F = M - TS = \frac{\Lambda(\varepsilon) r_+^2}{g(\varepsilon)^2 f(\varepsilon)} - \frac{G(\varepsilon) f(\varepsilon) q(\varepsilon)^2}{4} \left[\ln \left(\frac{r_+}{l(\varepsilon)} \right) - 1 \right]. \quad (30)$$

The heat capacity with fixed charge is given by

$$C_Q = T \frac{\left(\frac{\partial S}{\partial r_+} \right)_Q}{\left(\frac{\partial T}{\partial r_+} \right)_Q} = \frac{\pi r_+ [-2\Lambda(\varepsilon) r_+^2 + m(\varepsilon)^2 c(\varepsilon) c_1(\varepsilon) r_+ - 2f(\varepsilon)^2 g(\varepsilon)^2 G(\varepsilon) q(\varepsilon)^2]}{4g(\varepsilon) [-\Lambda(\varepsilon) r_+^2 + f(\varepsilon)^2 g(\varepsilon)^2 G(\varepsilon) q(\varepsilon)^2]}. \quad (31)$$

From the above expression, the effects of gravity's rainbow on specific heat can be seen easily.

2. Grand canonical ensemble

In the grand canonical ensemble with a fixed chemical potential (electric potential, U , in this case) associated with the charge, the Gibbs free energy for such black holes is given by

$$\begin{aligned} \mathbb{G} &= M - TS - \mu Q, \\ &= \frac{\Lambda(\varepsilon) r_+^2}{4g(\varepsilon)^2} \left(\frac{1}{f(\varepsilon)} - \frac{1}{2} \right) + \frac{m(\varepsilon)^2 c(\varepsilon) c_1(\varepsilon) r_+}{8g(\varepsilon)^2} \left(1 - \frac{1}{f(\varepsilon)} \right) \\ &\quad + \frac{1}{4} \left(\frac{\mu(\varepsilon)}{\ln \left(\frac{r_+}{l} \right)} \right)^2 f(\varepsilon) \left[G(\varepsilon) - \frac{f(\varepsilon)}{g(\varepsilon)^2} \ln \left(\frac{r_+}{l(\varepsilon)} \right) - 2G(\varepsilon) \ln \left(\frac{r_+}{l(\varepsilon)} \right) \right]. \end{aligned} \quad (32)$$

The Hawking temperature for the black holes in the grand canonical ensemble is given by

$$T = -\frac{\Lambda(\varepsilon) r_+}{2\pi f(\varepsilon) g(\varepsilon)} + \frac{m(\varepsilon)^2 c(\varepsilon) c_1(\varepsilon)}{4\pi f(\varepsilon) g(\varepsilon)} - \frac{f(\varepsilon) g(\varepsilon) G(\varepsilon)}{2\pi r_+} \left(\frac{\mu(\varepsilon)}{\ln \left(\frac{r_+}{l(\varepsilon)} \right)} \right)^2. \quad (33)$$

Now, the heat capacity with a fixed chemical potential is calculated by

$$C_\mu = T \left(\frac{\partial S}{\partial T} \right)_\mu = \frac{\pi^2 f(\varepsilon) g(\varepsilon) \left(\ln \left(\frac{r_+}{l(\varepsilon)} \right) \right)^3 r_+^2 T}{f(\varepsilon)^2 g(\varepsilon)^2 G(\varepsilon) \mu(\varepsilon)^2 \left(\ln \left(\frac{r_+}{l(\varepsilon)} \right) + 2 \right) - \Lambda(\varepsilon) r_+^2 \left(\ln \left(\frac{r_+}{l(\varepsilon)} \right) \right)^3}. \quad (34)$$

Next, we will study thermodynamical aspects of these black holes with the help of obtained thermodynamical quantities.

III. THERMODYNAMICAL ASPECTS OF CHARGED MASSIVE BTZ BLACK HOLES IN GRAVITY'S RAINBOW

In this section, we are interested to study, in particular, mass, temperature, heat capacity of the charged massive BTZ black holes in gravity's rainbow. We will discuss the free energy and phase diagram for such black holes as well.

A. Mass/Internal energy

Our first item of the interest is mass of black holes. The total mass of black holes has usually the interpretation of internal energy of a typical system. Evidently, the mass has three distinctive terms: cosmological constant term, $\Lambda(\varepsilon)$, massive term, $m(\varepsilon)$, and charge term, $q(\varepsilon)$. Depending on the choices of different values for these terms, the internal energy could have one of the following cases:

I) It is a positive definite function with a minimum. II) It is a positive definite function everywhere except at the a point which is an extreme root. III) It may have two roots with a region of negativity between these roots and a minimum. IV) Being only an increasing function of the horizon radius without any minimum.

Considering the positive nature of energy functions and other energy dependent constants, we find that the charge term contributes to negativity of the internal energy. As for $\Lambda(\varepsilon)$, its contribution depends on the type of spacetime we are working in. For anti-de Sitter spacetime, this term has constructive effects on the values of internal energy. Whereas, for de Sitter case, the internal energy is a decreasing function of this term. For the mass term, one finds that its effect depends on the choices of $c_1(\varepsilon)$. For negative values of this parameter, the mass term has negative effects on values of the internal energy while the opposite effect is true for positive $c_1(\varepsilon)$.

In general, it is not possible to obtain the root of internal energy analytically. But, regarding a vanishing term, it is possible to do so in which the results are given as

$$r_1|_{q(\varepsilon)=0} = \frac{m(\varepsilon)^2 c(\varepsilon) c_1(\varepsilon)}{\Lambda(\varepsilon)}, \quad (35)$$

$$r_2|_{m(\varepsilon)=0} = l(\varepsilon) \exp \left[-\frac{1}{2} \text{Lambert}W \left(\frac{\Lambda(\varepsilon) l(\varepsilon)^2}{G(\varepsilon) q(\varepsilon)^2 f(\varepsilon)^2 g(\varepsilon)^2} \right) \right], \quad (36)$$

$$r_3|_{\Lambda(\varepsilon)=0} = \frac{-2 G(\varepsilon) q(\varepsilon)^2 f(\varepsilon)^2 g(\varepsilon)^2 \text{Lambert}W \left(-\frac{m(\varepsilon)^2 c(\varepsilon) c_1(\varepsilon)}{2 G(\varepsilon) q(\varepsilon)^2 f(\varepsilon)^2 g(\varepsilon)^2} \right)}{m(\varepsilon)^2 c(\varepsilon) c_1(\varepsilon)}. \quad (37)$$

As for the high energy limit of internal energy, the dominant term is the charge term. In the absence of electric part of the solutions, for vanishing horizon radius, hence evaporation, the total mass of black holes would vanish too. Interestingly, it is possible to eliminate the effects of electric part by setting, $l(\varepsilon) = r_+$. Therefore, for including the effects of electric charge, the limit $l(\varepsilon) \neq r_+$ must be satisfied. The second dominant term after the electric charge is the massive term which highlights the effects of the massive gravity in high energy regime. On the other hand, for asymptotical behavior, the leading term will be the cosmological constant term. In the absence of this term, the asymptotical behavior of the system will be governed by the massive term which again, represents the effects of generalization to massive gravity. Considering these two cases, one can conclude that for medium black holes, the internal energy is highly affected by massive gravity. This means that for medium black holes, the effects of the presence of massive gravitons would be detectable within the internal energy. As for gravity's rainbow, except for the root of mass in the absence of electric charge, the obtained roots, high energy limit and asymptotical behavior are highly affected by generalization to gravity's rainbow. Consequently, the extracted properties and behaviors of the internal energy (existence of roots and negative values for internal energy) depend on the choices of rainbow functions of the metric, hence gravity's rainbow. Coupling different orders of rainbow functions with different parameters provides the possibility of manipulation of properties and behaviors of the internal energy.

B. Temperature

Now, let us focus on the temperature of these black holes. Here too, the charge term has negative contribution on the values of temperature. The effects of cosmological term depend on the spacetime under consideration. For adS black holes, the cosmological term has positive effects on temperature while for dS spacetime, the temperature is a decreasing function of this term. Interestingly, the massive term is not coupled with any order of horizon radius and it behaves as a constant. The roots of temperature are marking bound points. The reason for such naming is as follows; in classical thermodynamics, the negative values of temperature are interpreted as non-physical solutions.

Therefore, the roots of temperature separates physical solutions from non-physical ones. It is a matter of calculation to show that these black holes have following roots

$$r|_{T=0} = \frac{m(\varepsilon)^2 c(\varepsilon) c_1(\varepsilon) \pm \sqrt{m(\varepsilon)^4 c(\varepsilon)^2 c_1(\varepsilon)^2 - 16 \Lambda(\varepsilon) G(\varepsilon) q(\varepsilon)^2 f(\varepsilon)^2 g(\varepsilon)^2}}{4 \Lambda(\varepsilon)}. \quad (38)$$

The existence of real valued root for the temperature is limited to following condition

$$m(\varepsilon)^4 c(\varepsilon)^2 c_1(\varepsilon)^2 - 16 \Lambda(\varepsilon) G(\varepsilon) q(\varepsilon)^2 f(\varepsilon)^2 g(\varepsilon)^2 \geq 0, \quad (39)$$

which could be used to extract a specific limitation for the mass of graviton in term of other parameters

$$m(\varepsilon) = \left(\frac{16 \Lambda(\varepsilon) G(\varepsilon) q(\varepsilon)^2 f(\varepsilon)^2 g(\varepsilon)^2}{c(\varepsilon)^2 c_1(\varepsilon)^2} \right)^{\frac{1}{4}}. \quad (40)$$

Once more, we emphasize that by satisfying obtained condition, the roots of temperature will be real valued. For adS black holes, only one positive valued root exists for the temperature which is the negative branch of the obtained roots. On the contrary, for dS black holes, two positive valued roots may exist for the temperature. The existence of second positive valued root depends on the following condition

$$0 < \sqrt{m(\varepsilon)^4 c(\varepsilon)^2 c_1(\varepsilon)^2 - 16 \Lambda(\varepsilon) G(\varepsilon) q(\varepsilon)^2 f(\varepsilon)^2 g(\varepsilon)^2} < m(\varepsilon)^2 c(\varepsilon) c_1(\varepsilon), \quad (41)$$

which is partly similar to the condition for having real valued roots. The resulting roots show that the contributions of rainbow functions of the metric, could only be observed in the electric charge term. It means that in the roots of temperature, the energy functions of metric are only coupled with the electric charge term.

In the absence of electric charge, the root of temperature is given by

$$r_{T=0 \text{ with } q(\varepsilon)=0} = \frac{m(\varepsilon)^2 c(\varepsilon) c_1(\varepsilon)}{2 \Lambda(\varepsilon)}, \quad (42)$$

which shows that the positive valued roots exist only for dS spacetime and this root is an increasing function of the graviton's mass and a decreasing function of the cosmological constant. Interestingly, in this case, the rainbow functions have no effects on the root of temperature.

For massless gravitons case, the root of temperature will be obtained as

$$r_{T=0 \text{ with } m(\varepsilon)=0} = - \frac{\sqrt{-\Lambda(\varepsilon) G(\varepsilon) g(\varepsilon) f(\varepsilon) q(\varepsilon)}}{\Lambda(\varepsilon)}. \quad (43)$$

Evidently, for this case, the real valued root only exists for adS black holes and only one positive root could be extracted for these black holes in this case. Here, the root is an increasing function of the electric charge and rainbow functions.

For the absence of cosmological constant, the root of temperature could be calculated as

$$r_{T=0 \text{ with } \Lambda(\varepsilon)=0} = 2 \frac{G(\varepsilon) q(\varepsilon)^2 f(\varepsilon)^2 g(\varepsilon)^2}{m(\varepsilon)^2 c(\varepsilon) c_1(\varepsilon)}, \quad (44)$$

which, contrary to the absence of electric charge case, is a decreasing function of the graviton's mass. It is worthwhile to mention that in this case, the root is an increasing function of the electric charge and rainbow functions.

The high temperature limit of these black holes are governed by the electric charge term. On the other hand, the leading order in asymptotical behavior is the cosmological constant term. Interestingly, in the absence of electric charge (whether setting electric charge zero or consider the case $l(\varepsilon) = r_+$), the next leading order in high temperature will be the massive term. Now, remembering that this term in the temperature is not coupled with any order of the horizon radius, one can see that for the evaporation of black holes (vanishing horizon radius) the temperature will be non-zero. In other words, in the evaporation of these black holes, a trace of the existence of black holes is left behind which presents itself as fluctuation in the temperature of spacetime. This provides the possibility of the existence of black hole' information after their evaporation. This remnant, for the temperature of black holes, is an increasing function of the graviton's mass and a decreasing function of the rainbow functions. Considering the effects of massive gravity, one can state that thermodynamical behavior of temperature for medium black holes is governed by the mass of graviton. On the other hand, the effects of gravity's rainbow on the temperature could be observed for the three cases of small, medium and large black holes. In other words, the generalization of gravity's rainbow, contrary to massive gravity, has some effects on the temperature of all black holes with different sizes. It is worthwhile to mention that the effects of gravity's rainbow on the temperature of medium and large black holes are the same while for the small black holes, these effects are opposite.

C. Heat capacity

By taking a closer look at the heat capacity, one can see that its numerator is the same as temperature. Therefore, the roots of heat capacity and temperature are the same, and the arguments that were stated in the last subsection (for the temperature and its roots) can apply for the heat capacity and its roots as well. On the other hand, the denominator of heat capacity contains information regarding phase transition points. In other words, the divergence point of heat capacity (roots of denominator of the heat capacity) can be characterized as a phase transition.

Denominator of the heat capacity contains only electric charge and cosmological constant terms with coupling of rainbow functions and so it does not depend on the massive gravity parameter. In other words, the generalization to gravity's rainbow affects the divergencies of heat capacity, hence phase transitions of the black holes, whereas the existence of massive gravitons does not affect the phase transitions of these black holes. It is a matter of calculation to show that by using Eq. (31), the positive valued divergence point of heat capacity is obtained as

$$r_{c_Q \rightarrow \infty} = \frac{\sqrt{\Lambda(\varepsilon)G(\varepsilon)g(\varepsilon)f(\varepsilon)q(\varepsilon)}}{\Lambda(\varepsilon)}. \quad (45)$$

Evidently, the real valued phase transition points are observed for dS black holes only. The existence of the phase transition point depends on the electric charge and cosmological terms. This phase transition point is an increasing function of the electric charge and rainbow functions.

The high energy limit of the heat capacity is given by

$$\lim_{\text{very small } r_+} C_Q = -\frac{\pi}{2g(\varepsilon)} r_+ + \frac{\pi m(\varepsilon)^2 c(\varepsilon)c_1(\varepsilon)}{4g(\varepsilon)^3 G(\varepsilon)q(\varepsilon)^2 f(\varepsilon)^2} r_+^2 - \frac{\Lambda(\varepsilon)\pi}{g(\varepsilon)^3 G(\varepsilon)q(\varepsilon)^2 f(\varepsilon)^2} r_+^3 + O(r_+^4), \quad (46)$$

in which the effects of generalization to gravity's rainbow could be observed in dominant term. The presence of electric charge and massive gravity could be detected from second dominant term of the high energy limit. Here, the massive parameter and electric charge are coupled with each other. The contribution of the cosmological constant could be seen in third leading order, which is coupled with the electric charge as well. Taking a closer look, one can see that the effects of gravity's rainbow are presented in all three leading orders of high energy limit of the heat capacity. As one can see, the effects of massive gravity could be more highlighted for the medium black holes while for large black holes, the effect of cosmological constant governs the behavior of heat capacity.

Interestingly, in the absence of electric charge (similar to the cases that were studied in temperature), the dominant term for the high energy limit behaves like a constant including massive gravity and cosmological constant in following form

$$\lim_{\text{very small } r_+} C_Q = -\frac{\pi m(\varepsilon)^2 c(\varepsilon)c_1(\varepsilon)}{4g(\varepsilon)\Lambda(\varepsilon)} + \frac{\pi}{2g(\varepsilon)} r_+ + O(r_+^2), \quad \text{for } q(\varepsilon) = 0. \quad (47)$$

Similar to the temperature, here too, for vanishing horizon radius (evaporation of these black holes), the heat capacity will be non-zero. This shows that the traces of the existence of black holes after their evaporation could be observed in differences of heat capacity of the place where black holes existed. In this case, one can see that graviton's mass modify thermodynamical behavior of the black holes in their last stage of existence. The modifications of the gravity's rainbow could be seen by the presence of rainbow function, $g(\varepsilon)$.

On the other hand, for the asymptotic limit, one can derive following relation for the heat capacity

$$\lim_{\text{very large } r_+} C_Q = \frac{\pi}{2g(\varepsilon)} r_+ - \frac{\pi m(\varepsilon)^2 c(\varepsilon)c_1(\varepsilon)}{4g(\varepsilon)\Lambda(\varepsilon)} + \frac{G(\varepsilon)q(\varepsilon)^2 f(\varepsilon)^2 g(\varepsilon)\pi}{\Lambda(\varepsilon)r_+} + O(r_+^{-2}). \quad (48)$$

First of all, the dominant term for the asymptotical behavior of heat capacity is same as one extracted for the high energy limit with different sign. The second leading term in this case behaves like a constant term including massive gravity which, opposite to high energy limit, is coupled with the cosmological constant. The presence of electric charge part of the solutions could be observed in the third leading term of this case. Like previous, here, the presence of gravity's rainbow could be observed in all the three leading terms in asymptotical behavior of the heat capacity. One of the differences of this case with the high energy limit is the fact that in this limit, the presence of electric charge (cosmological constant) was only observed in denominator (numerator) of leading terms whereas, in the asymptotical case, the presence of electric charge (cosmological constant) was only observed in numerator (denominator) of the leading terms. In the absence of cosmological constant, the asymptotical behavior of heat capacity will be modified into

$$\lim_{\text{very large } r_+} C_Q = \frac{m(\varepsilon)^2 c(\varepsilon)c_1(\varepsilon)}{4g(\varepsilon)^3 G(\varepsilon)q(\varepsilon)^2 f(\varepsilon)^2} - \frac{\pi}{2g(\varepsilon)} r_+ \quad \text{for } \Lambda(\varepsilon) = 0, \quad (49)$$

which shows that the dominant term in the asymptotical behavior of heat capacity is a coupling between massive gravity and electric part of the solutions with the effects of gravity's rainbow. Considering the effects of massive gravity in both high energy regime and asymptotic limit, one can see that graviton's mass highly modifies the behavior of heat capacity in both of these regimes. This highlights the contribution of massive gravity in thermodynamical behavior of these black holes. The same could be also stated for gravity's rainbow due to the presence of rainbow functions in both of these regimes in the leading terms.

1. Free energy

The free energy contains information regarding the phase transition points. The chemical equilibrium is reached for a system when its free energy is minimized. In other words, the first order derivation of the free energy with respect to thermodynamical quantities vanishes at the equilibrium point. This equilibrium point marks the place where system goes under a phase transition. The other important information regarding the free energy is stored in its roots. In general, the root of free energy for these black holes is

$$r_{F=0} = l(\varepsilon) \exp \left[-\frac{1}{2} \text{Lambert}W \left(-\frac{\Lambda(\varepsilon)l(\varepsilon)^2 \exp(2)}{G(\varepsilon)q(\varepsilon)^2 f(\varepsilon)^2 g(\varepsilon)^2} \right) + 1 \right]. \quad (50)$$

Using the concept which was introduced for obtaining phase transition point from the free energy, one can show that the positive valued extremum point of the free energy is obtained as

$$r_{F \rightarrow \infty} = \frac{\sqrt{\Lambda(\varepsilon)G(\varepsilon)}g(\varepsilon)f(\varepsilon)q(\varepsilon)}{\Lambda(\varepsilon)}, \quad (51)$$

which is exactly the same phase transition point obtained for the heat capacity. Therefore, divergencies of the heat capacity and extremum of the free energy coincide with each other. The high energy limit of free energy is governed by the electric charge term. But here, similar to the case of internal energy, it is possible to cancel the effects of charge term through setting $l(\varepsilon) = r_+(\varepsilon)$. On the contrary, the leading order in the asymptotical behavior of free energy is the cosmological constant. In both of the mentioned regimes, the effects of gravity's rainbow could be observed by the coupling of energy functions with different parameters. It is worthwhile to mention that the free energy is independent of the generalization to massive gravity. In other words, the free energy (energy which could be converted to work) is independent of the graviton's mass.

Using the obtained extremum (critical horizon radius), one can obtain internal energy, temperature and free energy of the phase transition point as

$$M_{\text{Phase Transition}} = \frac{q(\varepsilon)}{8g(\varepsilon)\Lambda(\varepsilon)} \left(m(\varepsilon)^2 c(\varepsilon) c_1(\varepsilon) \sqrt{\Lambda(\varepsilon)G(\varepsilon)} - f(\varepsilon)G(\varepsilon)q(\varepsilon)g(\varepsilon)\Lambda(\varepsilon) - \left[1 + 2 \ln \left(\frac{\sqrt{\Lambda(\varepsilon)G(\varepsilon)}g(\varepsilon)f(\varepsilon)q(\varepsilon)}{\Lambda(\varepsilon)l(\varepsilon)} \right) \right] \right), \quad (52)$$

$$T_{\text{Phase Transition}} = \frac{m(\varepsilon)^2 c(\varepsilon) c_1(\varepsilon) \sqrt{\Lambda(\varepsilon)G(\varepsilon)} - 4 f(\varepsilon)G(\varepsilon)q(\varepsilon)g(\varepsilon)\Lambda(\varepsilon)}{4\pi f(\varepsilon)g(\varepsilon)\sqrt{\Lambda(\varepsilon)G(\varepsilon)}}, \quad (53)$$

$$F_{\text{Phase Transition}} = \frac{f(\varepsilon)G(\varepsilon)q(\varepsilon)^2}{8} \left[3 - 2 \ln \left(\frac{\sqrt{\Lambda(\varepsilon)G(\varepsilon)}g(\varepsilon)f(\varepsilon)q(\varepsilon)}{\Lambda(\varepsilon)l(\varepsilon)} \right) \right], \quad (54)$$

where we should regard the positive cosmological constant to obtain real valued quantities.

2. Phase diagrams

In order to complete our discussion regarding thermodynamical structure of these black holes, we have plotted a series of the diagrams for the mass/internal energy (Figs. 3 and 4), the temperature (Figs. 5 and 6), the heat capacity (Figs. 5 and 6) and the free energy (Fig. 7) for two cases of dS and adS spacetime. In Ref. [85], it was shown that in order to remove existence of ensemble dependency, $l(\varepsilon)$ which was inserted for the sake of dimensionless argument, should be replaced by following relation

$$\Lambda(\varepsilon) = \pm \frac{1}{l(\varepsilon)}, \quad (55)$$

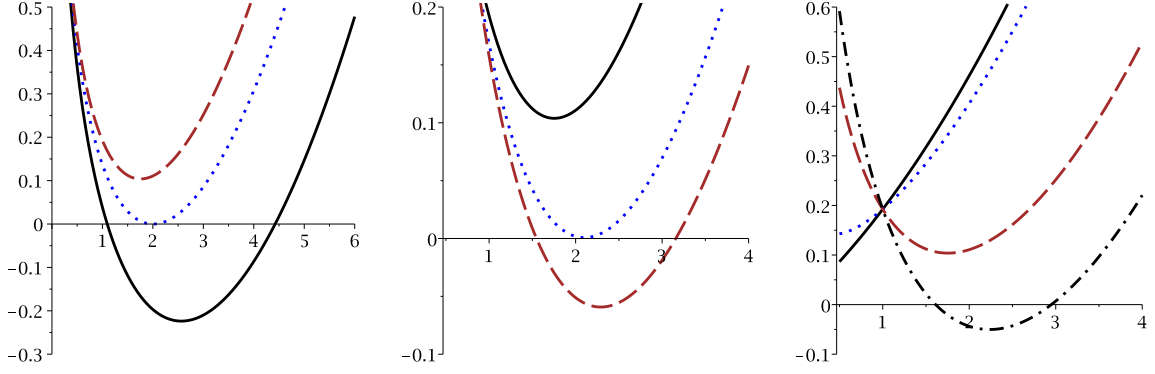


FIG. 3: For different scales: M versus r_+ for $G(\varepsilon) = l(\varepsilon) = 1$, $c(\varepsilon) = c_1(\varepsilon) = 2$, $g(\varepsilon) = 1.9$ and $\Lambda(\varepsilon) = -1$. Left panel: $q(\varepsilon) = 1.5$, $f(\varepsilon) = 0.9$, $m(\varepsilon) = 0$ (continuous line), $m(\varepsilon) = 0.8$ (dotted line) and $m(\varepsilon) = 1$ (dashed line). Middle panel: $q(\varepsilon) = 1.5$, $m(\varepsilon) = 1$, $f(\varepsilon) = 0.9$ (continuous line), $f(\varepsilon) = 1.03$ (dotted line) and $f(\varepsilon) = 1.1$ (dashed line). Right panel: $f(\varepsilon) = 0.9$, $m(\varepsilon) = 1$, $q(\varepsilon) = 0$ (continuous line), $q(\varepsilon) = 0.6$ (dotted line), $q(\varepsilon) = 1.5$ (dashed line) and $q(\varepsilon) = 1.8$ (dashed-dotted line).

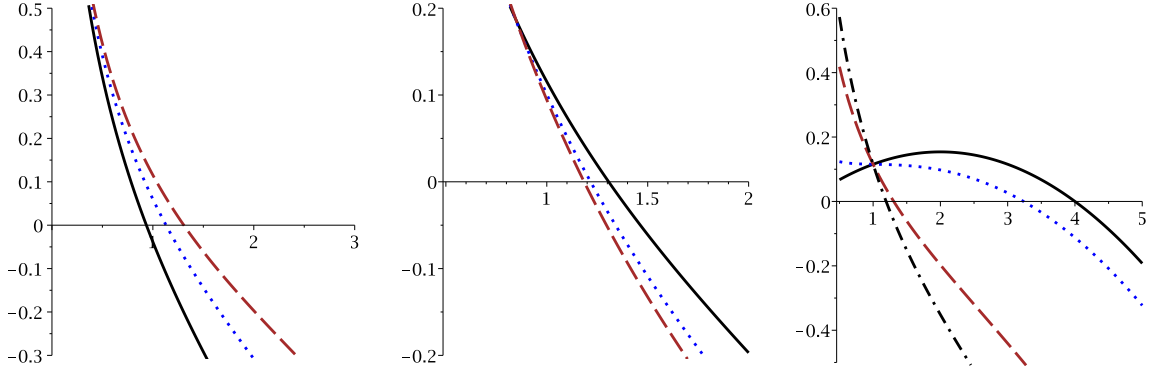


FIG. 4: For different scales: M versus r_+ for $G(\varepsilon) = l(\varepsilon) = 1$, $c(\varepsilon) = c_1(\varepsilon) = 2$, $g(\varepsilon) = 1.9$ and $\Lambda(\varepsilon) = 1$. Left panel: $q(\varepsilon) = 1.5$, $f(\varepsilon) = 0.9$, $m(\varepsilon) = 0$ (continuous line), $m(\varepsilon) = 0.8$ (dotted line) and $m(\varepsilon) = 1$ (dashed line). Middle panel: $q(\varepsilon) = 1.5$, $m(\varepsilon) = 1$, $f(\varepsilon) = 0.9$ (continuous line), $f(\varepsilon) = 1.03$ (dotted line) and $f(\varepsilon) = 1.1$ (dashed line). Right panel: $f(\varepsilon) = 0.9$, $m(\varepsilon) = 1$, $q(\varepsilon) = 0$ (continuous line), $q(\varepsilon) = 0.6$ (dotted line), $q(\varepsilon) = 1.5$ (dashed line) and $q(\varepsilon) = 1.8$ (dashed-dotted line).

in which positive branch is related to dS spacetime while the opposite is for AdS solutions. Hereafter, we use Eq. (55) to plot phase diagrams.

Studying mass/internal energy diagrams for adS black holes shows that depending on the choices of different parameters, this quantity could have a minimum. This minimum could have a negative mass/internal energy which indicates that two roots for this quantity exists with region of negative mass/internal energy. The exception is for the absence and small values of electric charge. For these cases, there exists a mass for the black holes in the limit of vanishing horizon radius (see Fig. 3 for more details).

As for dS case, the mass/internal energy is a decreasing function of the horizon radius with one root. The only exception is for the absence of electric charge. In this case, a maximum is formed for the mass/internal energy with one root (see Fig. 4 for more details). For dS black holes, the region of positivity of the mass/internal energy is located before root.

The temperature and heat capacity have the same roots. Before the root, for adS black holes, temperature and heat capacity are negative and solutions are non-physical. Whereas, after the root, both the temperature and the heat capacity are positive valued and solutions are physical and enjoy thermally stability. Interestingly, in the absence of electric charge, temperature and heat capacity are positive and non-zero for vanishing horizon radius, which confirms our earlier discussion regarding the existence of remnant for these quantities (see Fig. 5 for more details).

For dS black holes, the behaviors of temperature and heat capacity are completely different. Here, the temperature has one maximum. If the maximum is located at a positive temperature, the heat capacity (and temperature) will have two roots. Between these roots, only positive temperature hence, the physical solutions exists. Otherwise,

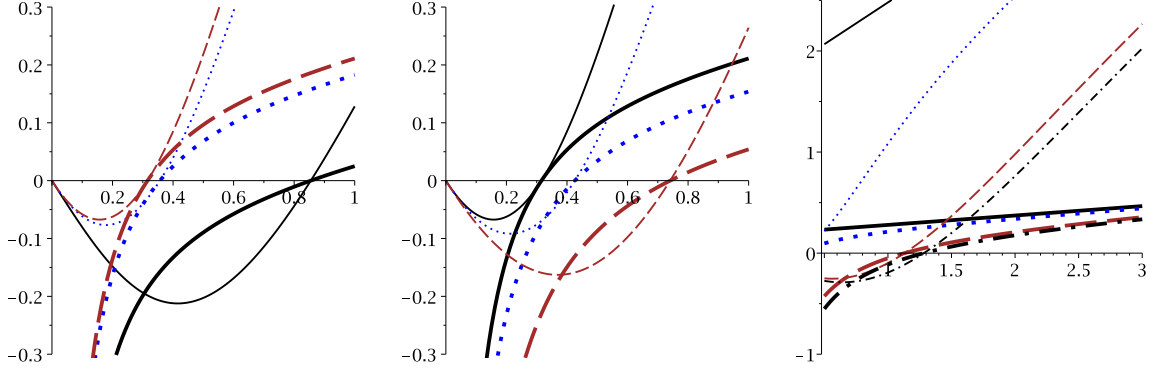


FIG. 5: For different scales: C_Q and T (bold lines) versus r_+ for $G(\varepsilon) = l(\varepsilon) = 1$, $c(\varepsilon) = c_1(\varepsilon) = 2$, $g(\varepsilon) = 1.9$ and $\Lambda(\varepsilon) = -1$. Left panel: $q(\varepsilon) = 0.5$, $f(\varepsilon) = 0.9$, $m(\varepsilon) = 0$ (continuous line), $m(\varepsilon) = 0.92$ (dotted line) and $m(\varepsilon) = 1$ (dashed line). Middle panel: $q(\varepsilon) = 0.5$, $m(\varepsilon) = 1$, $f(\varepsilon) = 0.9$ (continuous line), $f(\varepsilon) = 1.07$ (dotted line) and $f(\varepsilon) = 1.5$ (dashed line). Right panel: $f(\varepsilon) = 0.9$, $m(\varepsilon) = 1$, $q(\varepsilon) = 0$ (continuous line), $q(\varepsilon) = 0.5$ (dotted line), $q(\varepsilon) = 1.1$ (dashed line) and $q(\varepsilon) = 1.2$ (dashed-dotted line).

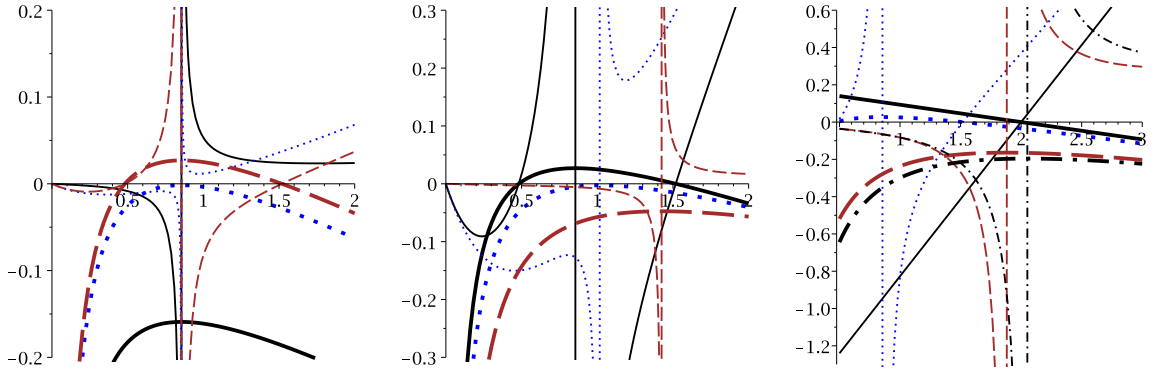


FIG. 6: For different scales: C_Q and T (bold lines) versus r_+ for $G(\varepsilon) = l(\varepsilon) = 1$, $c(\varepsilon) = c_1(\varepsilon) = 2$, $g(\varepsilon) = 1.9$ and $\Lambda(\varepsilon) = 1$. Left panel: $q(\varepsilon) = 0.5$, $f(\varepsilon) = 0.9$, $m(\varepsilon) = 0$ (continuous line), $m(\varepsilon) = 0.92$ (dotted line) and $m(\varepsilon) = 1$ (dashed line). Middle panel: $q(\varepsilon) = 0.5$, $m(\varepsilon) = 1$, $f(\varepsilon) = 0.9$ (continuous line), $f(\varepsilon) = 1.07$ (dotted line) and $f(\varepsilon) = 1.5$ (dashed line). Right panel: $f(\varepsilon) = 0.9$, $m(\varepsilon) = 1$, $q(\varepsilon) = 0$ (continuous line), $q(\varepsilon) = 0.5$ (dotted line), $q(\varepsilon) = 1.1$ (dashed line) and $q(\varepsilon) = 1.2$ (dashed-dotted line).

temperature is negative and the solutions are non-physical. At the maximum of temperature, the heat capacity acquires divergency which marks a phase transition point. The phase transition is between large black holes to small ones. This shows that thermally stable black holes only exist between smaller root and divergence point. The only exception for the presence of maximum in temperature is the absence of electric charge. In this case, temperature is a decreasing function of the horizon radius with one root. In positive region of the temperature, the heat capacity is negative. Therefore, for this case, the solutions are thermally unstable (see Fig. 6 for more details).

Finally, for adS case, the free energy is only a decreasing function of the horizon radius. The only exception for this case is for the absence of electric charge, where the free energy is negative valued without any root (see up panels of Fig. 5 for more details). For the dS case, a minimum is formed for the free energy. This minimum represents the point in which black holes go under second order phase transition. The only exception for this case is for the absence of electric charge. In this case, the free energy is an increasing function of the horizon radius (see down panels of Fig. 7 for more details).

IV. GEOMETRICAL THERMODYNAMICS

In this section, we employ GTs approach to investigate thermodynamical properties of the black holes by using the so-called HPEM metric. Applying GTs approach, we can extract some information regarding thermodynamical behavior of the system by studying the Ricci scalar of constructed phase space. In this method, the phase transition

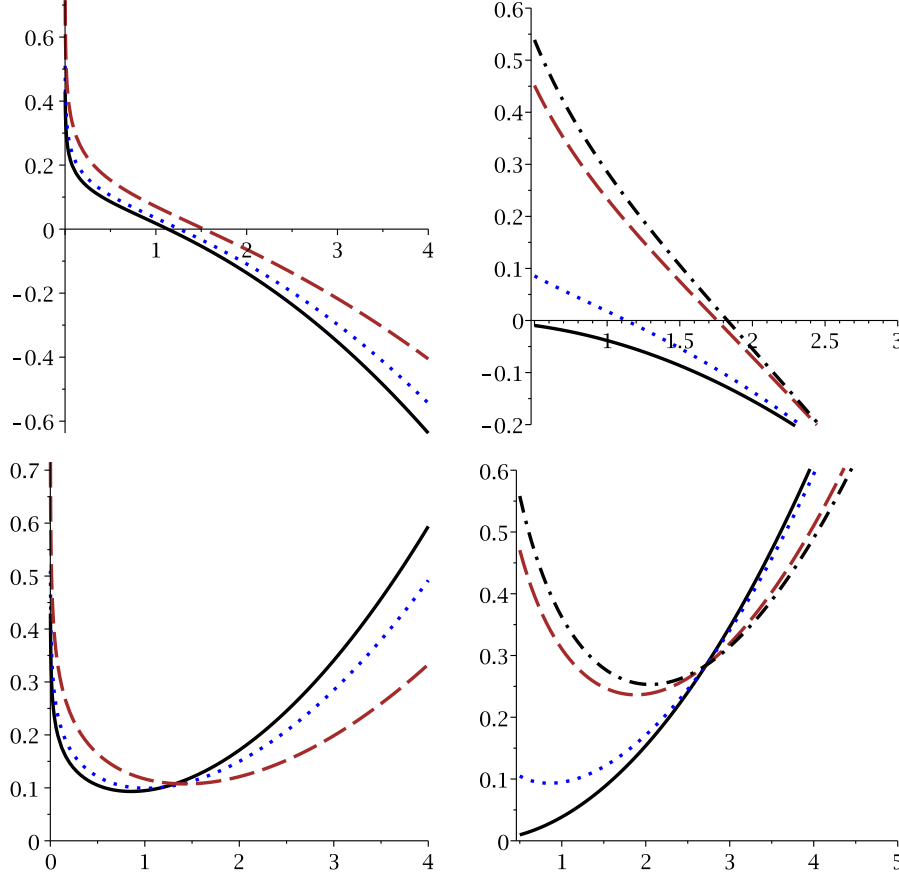


FIG. 7: For different scales: F versus r_+ for $G(\varepsilon) = l(\varepsilon) = 1$ and $g(\varepsilon) = 1.9$.

Left panel: $q(\varepsilon) = 0.5$, $f(\varepsilon) = 0.9$ (continuous line), $f(\varepsilon) = 1.07$ (dotted line) and $f(\varepsilon) = 1.9$ (dashed line).

Right panel: $f(\varepsilon) = 0.9$, $q(\varepsilon) = 0$ (continuous line), $q(\varepsilon) = 0.5$ (dotted line), $q(\varepsilon) = 1.1$ (dashed line) and $q(\varepsilon) = 1.2$ (dashed-dotted line).

Up panels: $\Lambda(\varepsilon) = -1$; Down panels: $\Lambda(\varepsilon) = 1$;

and bound points should be represented as divergencies of the Ricci scalar. Recent studies in the context of the GTs approaches for the black hole thermodynamics have shown that the Ricci scalars of Weinhold, Ruppeiner and Quevedo metrics may lead to extra divergencies which are not matched with the bound points and the phase transitions [72–75]. In other words, there were cases of mismatch between divergencies of the Ricci scalar and the mentioned points (bound and phase transition points), and also existence of extra divergency unrelated to these points were reported [72–75]. In order to overcome the shortcomings of the mentioned methods (Weinhold, Ruppeiner and Quevedo metrics), the HPEM method was introduced and it was shown that the specific structure of this metric provides satisfactory results regarding GTs of different classes of the black holes. In addition, this metric contains information which enables one to determine the type of divergencies to distinguish the divergencies related to the bound points and those correspond to the phase transition points.

The HPEM metric of a charged black hole is in the following form

$$ds^2 = S \frac{M_S}{M_{QQ}^3} (-M_{SS} dS^2 + M_{QQ} dQ^2), \quad (56)$$

in which $M_X = \partial M / \partial X$ and $M_{XX} = \partial^2 M / \partial X^2$. The denominator of Ricci scalar of this phase space is [72]

$$\text{denom}(\mathcal{R}) = 2S^3 M_{SS}^2 M_S^3. \quad (57)$$

In order to have consistent results, we employ HPEM metric and present table I and plot following two diagrams (Figs. 8 and 9).

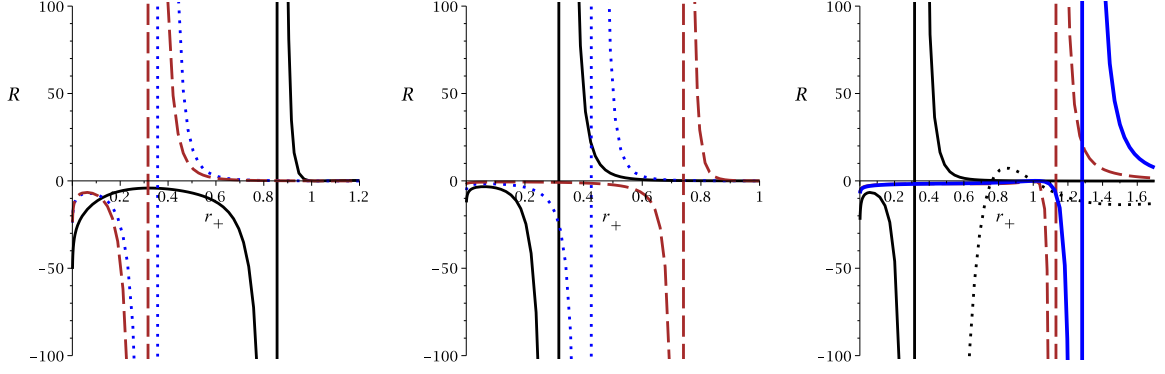


FIG. 8: For different scales: R (Ricci scalar) versus r_+ for $G(\varepsilon) = l(\varepsilon) = 1$, $c(\varepsilon) = c_1(\varepsilon) = 2$, $g(\varepsilon) = 1.9$ and $\Lambda(\varepsilon) = -1$. Left panel: $q(\varepsilon) = 0.5$, $f(\varepsilon) = 0.9$, $m(\varepsilon) = 0$ (continuous line), $m(\varepsilon) = 0.92$ (dotted line) and $m(\varepsilon) = 1$ (dashed line). Middle panel: $q(\varepsilon) = 0.5$, $m(\varepsilon) = 1$, $f(\varepsilon) = 0.9$ (continuous line), $f(\varepsilon) = 1.07$ (dotted line) and $f(\varepsilon) = 1.5$ (dashed line). Right panel: $f(\varepsilon) = 0.9$, $m(\varepsilon) = 1$, $q(\varepsilon) = 0$ (continuous line), $q(\varepsilon) = 0.5$ (dotted line), $q(\varepsilon) = 1.1$ (dashed line) and $q(\varepsilon) = 1.2$ (dashed-dotted line).

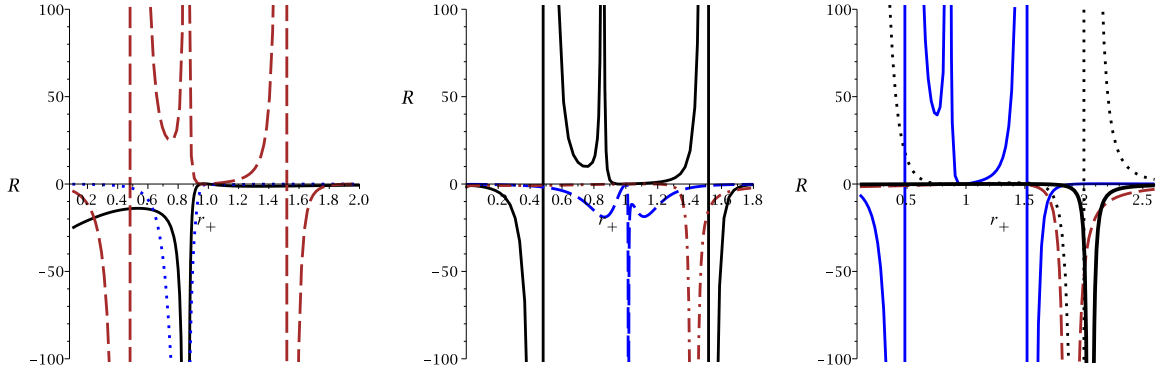


FIG. 9: For different scales: R (Ricci scalar) versus r_+ for $G(\varepsilon) = l(\varepsilon) = 1$, $c(\varepsilon) = c_1(\varepsilon) = 2$, $g(\varepsilon) = 1.9$ and $\Lambda(\varepsilon) = -1$. Left panel: $q(\varepsilon) = 0.5$, $f(\varepsilon) = 0.9$, $m(\varepsilon) = 0$ (continuous line), $m(\varepsilon) = 0.92$ (dotted line) and $m(\varepsilon) = 1$ (dashed line). Middle panel: $q(\varepsilon) = 0.5$, $m(\varepsilon) = 1$, $f(\varepsilon) = 0.9$ (continuous line), $f(\varepsilon) = 1.07$ (dotted line) and $f(\varepsilon) = 1.5$ (dashed line). Right panel: $f(\varepsilon) = 0.9$, $m(\varepsilon) = 1$, $q(\varepsilon) = 0$ (continuous line), $q(\varepsilon) = 0.5$ (dotted line), $q(\varepsilon) = 1.1$ (dashed line) and $q(\varepsilon) = 1.2$ (dashed-dotted line).

In the table I, R_∞ and C_∞ are, respectively, divergencies of the Ricci scalar and heat capacity. Also, C_0 and T_0 are the roots of heat capacity and temperature, respectively. Comparison between Figs. 8 and 9 with plotted diagrams in Figs. 5 and 6 (or see the table I, for more details), shows that all the bound and phase transition points are matched with divergencies of the Ricci scalar of the HPEM metric for different parameters. These coincidences between the divergencies of the Ricci scalar of HPEM metric with the bound and phase transition points of the heat capacity and temperature, confirm the validity of the results of this thermodynamical metric. So, one can use this method as an independent approach regarding studying thermodynamical properties of the black holes. Another interesting property of HPEM metric is related to the sign of Ricci scalar of the HPEM metric around the bound and phase transition points which depends on the type of point. As one can see, the sign of Ricci scalar around the bound point changes, while for phase transition point, the sign of Ricci scalar of the HPEM metric does not change (see Figs. 8 and 9, for more details). Therefore, by studying the sign of Ricci scalar of HPEM metric, we can characterize the type of divergencies. On the other hand, in GTs, the sign of Ricci scalar determines whether system has attractive (for negative sign) or repulsive (for positive sign) interaction around the bound and phase transition point. Here, we see that before the bound point, system has repulsive interaction and by crossing the bound point, the interaction is changed into attractive (see Figs. 8 and 9, for more details). On the other hand, for the phase transition point, the sign of Ricci scalar is positive (see Figs. 8 and 9, for more details). It is notable that, we can not extract these information by using the temperature and heat capacity of system. Therefore, we see that employing the HPEM metric provides extra information regarding the nature of interactions around the bound and phase transition points.

| $\Lambda(\varepsilon)$ | $f(\varepsilon)$ | $m(\varepsilon)$ | $q(\varepsilon)$ | R_∞ | C_∞ | C_0 | T_0 |
|------------------------|------------------|------------------|------------------|-----------------------------|------------|--------------------|--------------------|
| -1 | 0.9 | 0 | 0.5 | 0.85500 | — | 0.85500 | 0.85500 |
| -1 | 0.9 | 0.92 | 0.5 | 0.35668 | — | 0.35668 | 0.35668 |
| -1 | 0.9 | 1 | 0.5 | 0.31568 | — | 0.31568 | 0.31568 |
| -1 | 0.9 | 1 | 0.5 | 0.31568 | — | 0.31568 | 0.31568 |
| -1 | 1.07 | 1 | 0.5 | 0.42592 | — | 0.42592 | 0.42592 |
| -1 | 1.5 | 1 | 0.5 | 0.74086 | — | 0.74086 | 0.74086 |
| -1 | 0.9 | 1 | 0 | — | — | — | — |
| -1 | 0.9 | 1 | 0.5 | 0.31568 | — | 0.31568 | 0.31568 |
| -1 | 0.9 | 1 | 1.1 | 1.13029 | — | 1.13029 | 1.13029 |
| -1 | 0.9 | 1 | 1.2 | 1.28269 | — | 1.28269 | 1.28269 |
| 1 | 0.9 | 0 | 0.5 | 0.85500 | 0.85500 | — | — |
| 1 | 0.9 | 0.92 | 0.5 | 0.85500 | 0.85500 | — | — |
| 1 | 0.9 | 1 | 0.5 | (0.48137, 0.85500, 1.51862) | 0.85500 | (0.48137, 1.51862) | (0.48137, 1.51862) |
| 1 | 0.9 | 1 | 0.5 | (0.48137, 0.85500, 1.51862) | 0.85500 | (0.48137, 1.51862) | (0.48137, 1.51862) |
| 1 | 1.07 | 1 | 0.5 | 1.01650 | 1.01650 | — | — |
| 1 | 1.5 | 1 | 0.5 | 1.42500 | 1.42500 | — | — |
| 1 | 0.9 | 1 | 0 | 2.00000 | — | 2.00000 | 2.00000 |
| 1 | 0.9 | 1 | 0.5 | (0.48137, 0.85500, 1.51862) | 0.85500 | (0.48137, 1.51862) | (0.48137, 1.51862) |
| 1 | 0.9 | 1 | 1.1 | 1.88100 | 1.88100 | — | — |
| 1 | 0.9 | 1 | 1.2 | 2.05200 | 2.05200 | — | — |

TABLE I: Roots and divergencies of the Ricci scalar, heat capacity and temperature for $G(\varepsilon) = l(\varepsilon) = 1$, $g(\varepsilon) = 1.9$ and $c(\varepsilon) = c_1(\varepsilon) = 2$.

V. CONCLUSIONS

In this paper, we have considered the three dimensional black holes in the presence of massive gravity's rainbow. The solutions were extracted and the effects of massive gravity and gravity's rainbow on the geometrical structure of the black holes were studied.

Furthermore, the conserved and thermodynamical properties of these black holes were extracted in both canonical and grand canonical ensembles. It was shown that the existence of massive gravity and generalization of the gravity's rainbow modify thermodynamical quantities of the black holes. The effects of these generalizations were studied for different thermodynamical quantities.

It was shown that by suitable choices of different parameters, it is possible to obtain a minimum for the mass. This minimum could be located at negative values of the mass/internal energy which leads to the existence of a region of negativity for the mass/internal energy and two roots. Existence of the negative mass/internal energy is not valid in the classical thermodynamics of the black holes. Therefore, one can conclude that for this region of the negative mass/internal energy, black hole solutions do not exist. This puts a limitation on the values that different parameters can acquire.

Next, the temperature was taken into account and it was pointed out that one can acquire a root for this quantity. This root separates the physical solutions with positive temperature from non-physical ones with negative temperature. Remarkably, we have observed that the existence of root was restricted with satisfaction of a condition which was depending on the values of different parameters. In addition, it was shown that for a specific limit, there may exist a remnant of temperature for these black holes. In other words, after the evaporation of black holes, there will be traces of existence of these black holes in form of fluctuation in temperature of the spacetime.

Subsequently, the heat capacity was investigated and the possibility of divergence point was pointed out. This divergency marks the phase transition point for these black holes. The existence of real valued positive divergence point depends on the spacetime being dS or adS. The asymptotic and high energy limits of the heat capacity were studied as well and it was shown that these limits were governed by the gravity's rainbow and massive gravity generalizations.

After that, the free energy was studied and its root and divergence point were extracted. It was shown that the extremum of free energy and divergence point of the heat capacity are matched. The resulting critical horizon radius was used to obtain the critical temperature, the mass/internal energy and the free energy.

At last, we have used HPEM metric in the context GTs in order to study thermodynamical structure of these black holes. It was shown that this metric can describe the (non)physical and phase transition points of these black holes. Besides, by studying the sign of HPEM metric around the bound and phase transition points, it was possible to distinguish repulsive and attractive interactions of thermodynamical system.

The rainbow functions of the metric are originated from quantum corrections. Study conducted in this paper showed significant modifications in thermodynamics of the black holes with consideration of the such corrections. In other words, it was shown that in semi-classical/quantum regime, thermodynamics of the black holes would be modified into a level which differs from classical case. We observed that different orders of the rainbow functions affect the high energy and asymptotical behaviors of the solutions and their leading terms. On the other hand, we found that the highest contribution of massive gravitons could be observed in medium black holes. The only case, in which the effects of massive gravity could be observed for small (large) black holes was in the case of vanishing electric charge (cosmological constant). Obtained results here could be employed to study lattice like behavior in context of AdS/QCD correspondence. In addition, it is possible to employ the results of this paper to investigate the entropy spectrum and quasinormal modes. In fact, it would be interesting to investigate the effects of energy functions on these quantities. The results of this paper could also be employed in the context of AdS/CFT correspondence and central charges, specially considering the energy dependency of the constants and their effects on calculation of the central charges.

Acknowledgments

We thank Shiraz University Research Council. This work has been supported financially by the Research Institute for Astronomy and Astrophysics of Maragha, Iran.

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